Working Mathematically with Numbers

Heather McMaster & Michael Mitchelmore
Macquarie University

In previous articles, we have written about a W*orking Mathematically* approach to teaching the Space and Geometry, Measurement, Data, and Patterns and Algebra strands of the Stage 4 syllabus. In this article, we shall focus on the Number strand.

The NSW syllabus document states that "Numbers, in their various forms, are used to quantify and describe the world." (Board of Studies, 2002, p.46). An understanding of numbers and relationships between numbers is therefore fundamental to every other strand.

Many students struggle when with working with numbers because they have never developed a strong sense of their relative magnitudes. Students are better able to "see" patterns and relationships between numbers if they can visualise them in some way. Graphics scaffold students' understanding by making numbers "visible". Graphics are difficult and time-consuming for teachers to draw on a blackboard, and writers of mathematics textbooks need to limit their use because they take up valuable space. Since the advent of computers and data projectors however, these restrictions no longer exist.

In our workbooks *Working Mathematically: Number Stage 4* we have endeavoured to illustrate each concept using graphics that should be already familiar to students: the hundreds chart, the number line, scales, divided bar and sector graphs, column graphs and line graphs. For each concept, we use the same type of graphic to illustrate the same mathematical structure evident in a variety of contexts. As well as helping students to abstract the mathematics from the contexts, using the same graphic underscores connections between topics and between the various strands of the mathematics syllabus. In this paper, we will present examples of the types of graphics we use to illustrate concepts within the Stage 4 Number strand and the sequence in which we use them.

The decimal number system

We begin by comparing ancient numeral systems to our own so students can appreciate the importance of place value. This naturally leads to teaching about decimal place values. Because we use the metric system of measurement in this country, measurement scales are a useful graphic for teaching decimals. A scale is effectively the same as a number line except that most division points are generally not labelled. Discussing the labelling of these points helps students grasp ideas of the relative size of decimal numbers and the significance of each digit. It also prepares students for later when decimals will be related to fractions other than tenths, hundredths and thousandths.

Directed numbers

Students need to understand three meanings of the minus sign: "subtract", "negative" and "do the opposite" (or undo). We show negative numbers on a number line, show subtraction by moves to the left along the number line, and show "do the opposite" by changing a subtraction to an addition or changing an addition to a subtraction. Illustrating multiplication by negative numbers is more difficult. We do this by extending patterns of multiplication seen in column graphs like the ones shown in Figure 1 for the 4 times table and the (-4) times table. A line ruled through the tops of the columns shows the result of the multiplication for any real number.

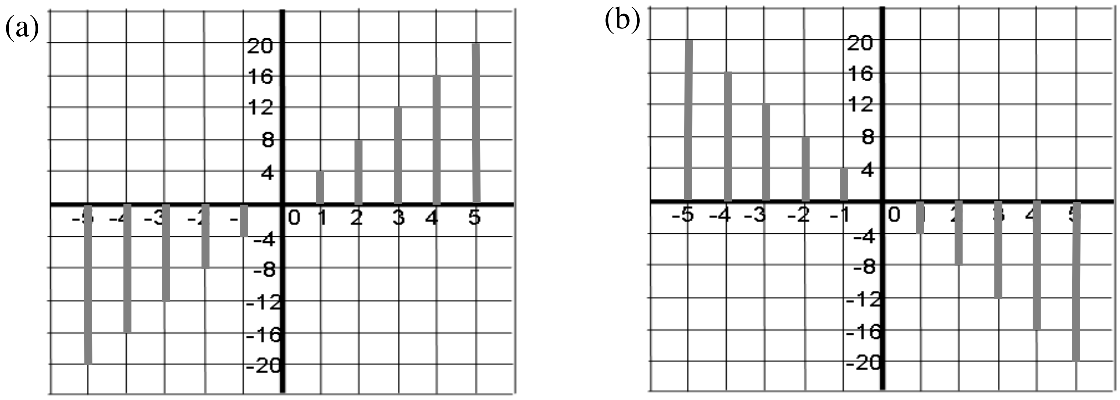


Figure 1: The results of integers being multiplied (a) by 4 and (b) by -4.

Factors and multiples

The colouring of sets of multiples on a hundreds chart is an easy and enjoyable way of finding lowest common multiples and highest common factors. The lengths of the rows are chosen so patterns for particular multiples (eg. multiples of 5 and 7) can se seen more clearly (Figure 2).

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96

One sand timer measures 5 minutes.
So by turning it over and over again, it measures multiples of 5 minutes.

Lightly shade all the boxes in this number chart that have multiples of 5 written in them.

Another timer measures multiples of 7 minutes.

Use a different type of shading to shade all the boxes in this chart that have multiples of 7 written in them.

Circle all the numbers in the chart that have been shaded twice.

Figure 2: An activity using a version of the hundreds chart.

A hundreds chart can also be used as a game board as in the game “Checkmaths” (Davis, 1968). In our workbooks, we use a hexagonal variation of a hundreds chart as a game board for “Divisibility Hex”. Games like these encourage instant recall of multiplication facts as well as strategic thinking. Without this instant recall, students are unable to recognise equivalent fractions – an essential skill for operating with fractions.

Fractions and percentages

We make equivalent fractions visible by asking students to divide up bars of equal length. By measuring the length of a bar, dividing by the denominator to get equally sized pieces and then counting the number of pieces according to the value of the numerator, students can see equivalent fractions (Figure 3). A bar divided into 100 equal pieces enables students to make the same visual link with percentages. These visual linkages support the calculations (the % sign simply meaning "out of 100").

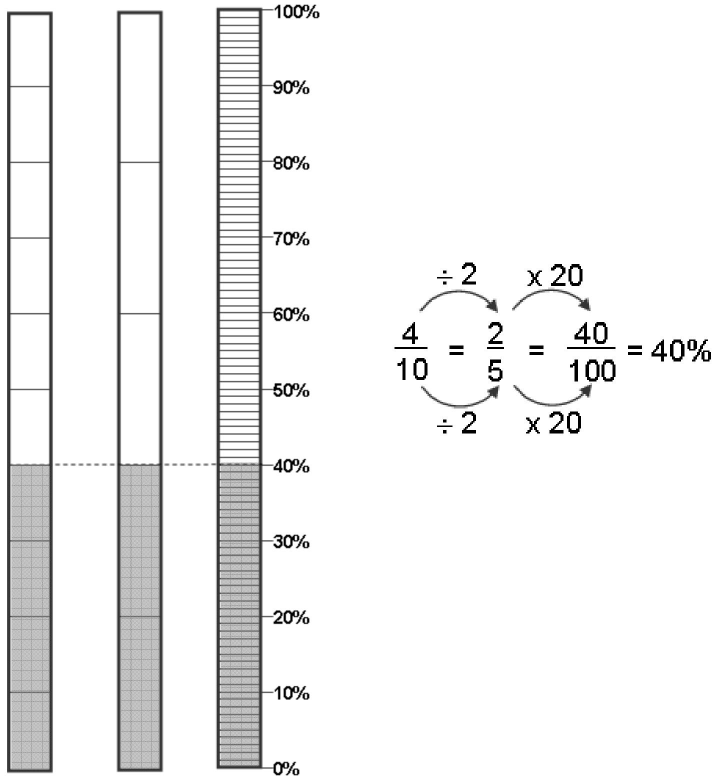


Figure 3: Divided bars used to link equivalent fractions and percentages

We later use divided bars and scales to illustrate operations with fractions (Figure 4).

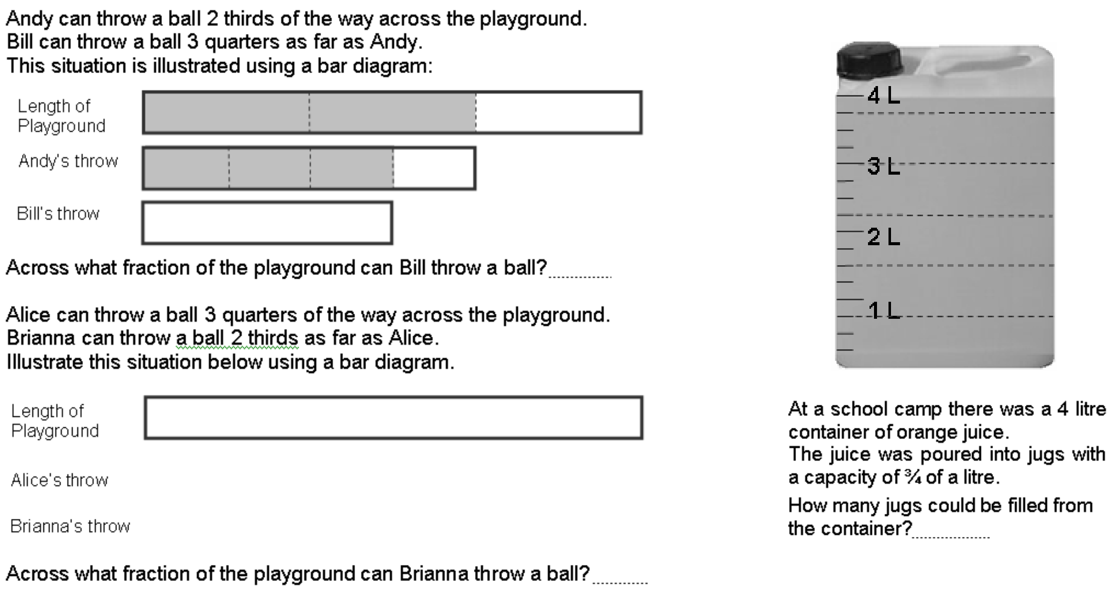


Figure 4: Operations with fractions using divided bars and scales

In Stage 3 and then on into Stage 4 there is an increasing emphasis on relative comparisons. Fractions are the basic tool needed to make relative comparisons and percentages are the most common form in which relative comparisons are expressed in everyday life.

Students soon learn to make a distinction between an *absolute comparison* (an additive relationship) and a *relative comparison* (a multiplicative relationship) when each context is coupled with a column graph like the one in Figure 5. From this graph of average life spans, students are asked which they would say is older, a 50 year old man or a 15 year old horse. The man is older in an absolute sense because 50 is greater than 15, but the horse is older in a relative sense because 15 out of 20 (¾ or 75%) is greater than 50 out of 80 (⅝ or 62.5%). This can be seen as the proportion of the man's column reached by the man compared to the proportion of the horse's column reached by the horse.

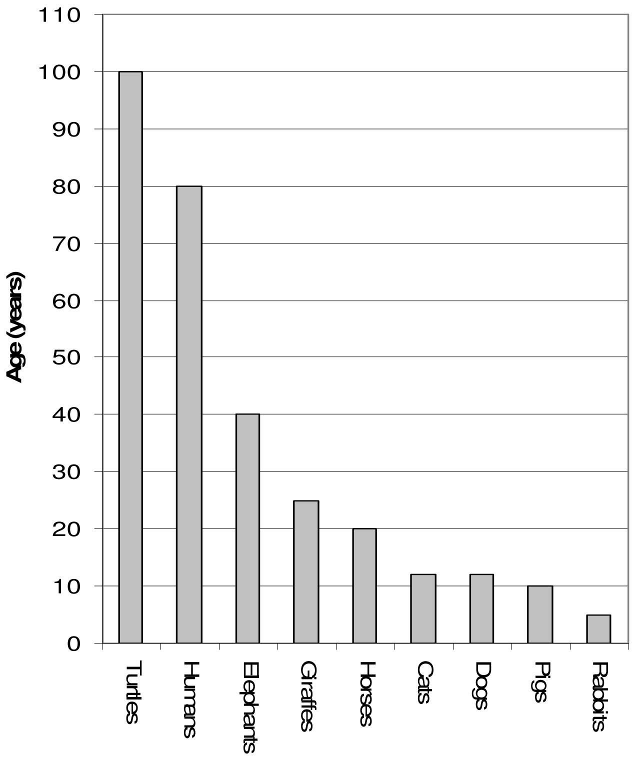


Figure 5: Average life spans

To extend the concept of a fraction to improper and mixed fractions and to extend the concept of a percentage to values greater than 100%, the bar model is replaced by a number of circles of equal size. Circles are more difficult than bars to divide into equal parts, but are helpful in that they better illustrate wholes. We therefore use them to illustrate complementary fractions and mixed numerals (Figure 6).

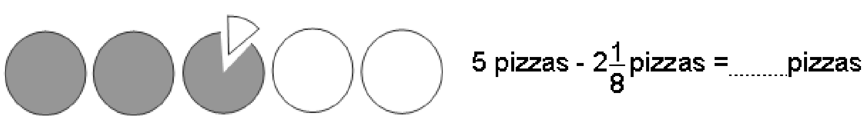


Figure 6: Subtraction of a mixed numeral using sectors.

Mixed and improper fractions are then placed on the number line. The number line also enables the positioning of negative fractions.

Rates and ratios

Rates and ratios are relative comparisons. In most textbooks, ratios are taught before rates. We begin with rates because students often find rates somewhat easier than ratios. Students are already familiar with rates in Stage 3 (eg. in travel graphs) and the different units in which each quantity is measured enable students to clearly distinguish the two quantities. They often have difficulty handling the change of units in rates, but if equivalent fractions are understood, this should not be a problem - the calculations being identical to those employed to find equivalent fractions.

By plotting data from a number of equivalent rates, students can see that the graph relating the two quantities is a straight line through the origin (Figure 7). The reason lies in the fact that the rate is constant.

Mass of aluminium (g)	2.7	5.4	8.1
Volume of aluminium (cm³)	1.0	2.0	3.0

x 2: $\frac{2.7}{1.0} = \frac{5.4}{2.0}$ x 2; x 3: $\frac{2.7}{1.0} = \frac{8.1}{3.0}$ x 3; x 1.5: $\frac{5.4}{2.0} = \frac{8.1}{3.0}$ x 1.5

Mass of water (g)	1.0	2.0	3.0
Volume of water (cm³)	1.0	2.0	3.0

x 2: $\frac{1.0}{1.0} = \frac{2.0}{2.0}$ x 2; x 3: $\frac{1.0}{1.0} = \frac{3.0}{3.0}$ x 3; x 1.5: $\frac{2.0}{2.0} = \frac{3.0}{3.0}$ x 1.5

Mass of cork (g)	1.0	2.0	3.0
Volume of cork (cm³)	1.0	0.25	0.5

x 2: $\frac{0.25}{1.0} = \frac{0.5}{2.0}$ x 2; x 3: $\frac{0.25}{1.0} = \frac{0.75}{3.0}$ x 3; x 1.5: $\frac{0.5}{2.0} = \frac{0.75}{3.0}$ x 1.5

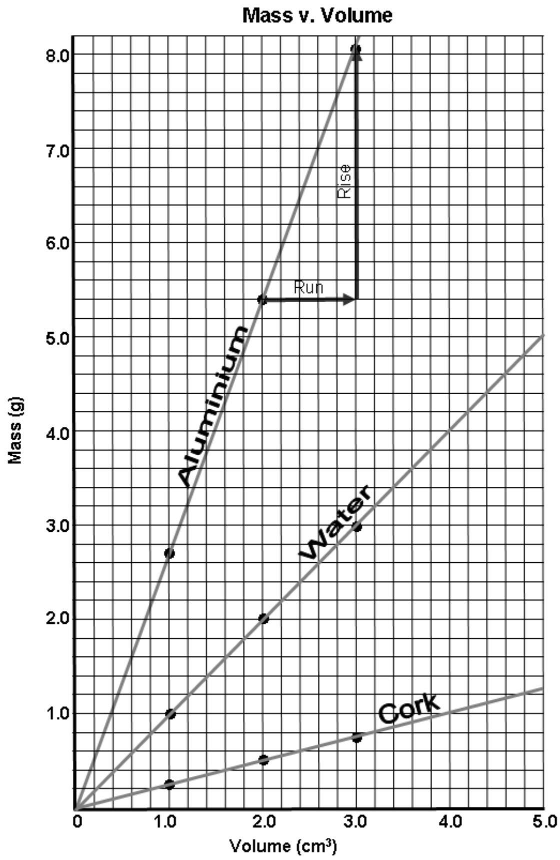


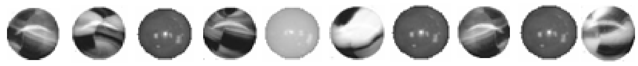
Figure 7: Tables and line graphs showing equivalent rates.

At this point, the concept of a gradient of a line in a graph is introduced. A gradient shows how one variable changes in relation to another. If a rate is constant, the gradient of the line on the graph is constant. So the gradient measured between different points on the same line gives rates that are equivalent.

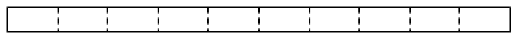
A ratio is a special type of rate. It is a rate that can be written without the units after the numbers. So like a rate, a ratio is also illustrated as the gradient of a line on a graph.

It is important for students to understand why rates and ratios "behave" like fractions, even though they are different. Whereas fractions relate the size of a part to the whole, ratios tend to relate the sizes of different parts. We use bar diagrams accompanying ratio contexts to illustrate this connection (Figure 8).

There are two types of marbles in this picture.



Shade the bar below to show the number of these marbles that are rainbow-coloured. Label the shaded part "rainbow".



What does the unshaded part of the bar represent? ....

Label the unshaded part of the bar "plain".

Write (in simplest form) the fraction of the marbles that are rainbow: ....

What is the simplest ratio of rainbow marbles to plain marbles? ....

How does the bar diagram show a fraction? ....

How does the bar diagram show a ratio? ....

Figure 8: An activity connecting fractions and ratios.

Probability

Probability is yet another type of relative comparison. If students are familiar with spinners, a visual connection can easily be made between the probability of a getting a particular colour on the spinner (the area of that coloured sector or sectors in relation to the area of the circle) and fractions. Because they can express a fraction as a percentage or a decimal or a ratio, they can also write a probability in any of these forms. A connection is also made between complementary fractions and the probabilities of complementary events.

Conclusion

A clear understanding of relative comparisons and the related multiplicative calculations is fundamental to a student's progress through the Stage 4 (decimals, fractions, percentages, rates, gradients, ratios, probability, relative frequency, scale factors, similarity) and beyond. We believe that for most students this understanding can be more readily achieved when calculations are consistently accompanied by familiar graphics.

It is also important, of course, to use the same types of graphics to investigate absolute comparisons which rely on additive calculations, as well as topics (such as numeration and percentage change) that involve both additive and multiplicative components. An intuition for the difference between multiplicative and additive structures may help students avoid such common errors as $\frac{2}{3}+\frac{3}{4}=\frac{2+3}{3+4}$ and $\sqrt{3^2+4^2}=3+4$.

Further examples of activities designed to involve students in mastering the Stage 4 Number strand may be found at www.workingmaths.net or on the 2008 Conference page of the MANSW website. In 2009, schools will be able to obtain a license to place digital files of all the workbooks in our *Working Mathematically* series on their school server.

References

Board of Studies (2002). *Mathematics Years 7-10 Syllabus.* Sydney: Author.

Davis, B (1968). *Checkmaths* [educational game]. Sydney:

McMaster, H., & Mitchelmore, M. (2008). *Working Mathematically: Number Stage 4* (Part A and Part B). Sydney: Workingmaths.