Working Mathematically in Patterns and Algebra (Part 2)

Michael Mitchelmore & Heather McMaster
Macquarie University

In the first part of this article (McMaster & Mitchelmore, 2007), we showed how algebra could be introduced to students through generalising patterns in familiar relationships. In this article, we show how this approach of generalising from realistic contexts may be continued in the second part of Stage 4.

Algebra is a difficult topic for many students. They must learn that they can deal with numbers even when they do not know what they are. In particular, they need to develop a sound concept of a *variable.* We believe that it is worth spending the first part of Stage 4 helping them to do this—showing how variables can be used to express relationships and giving students a strong understanding of tabular, graphical and symbolic representations. In doing so, there are many opportunities for Working Mathematically.

We believe that the main aim of the second part of Stage 4 should be to help students make generalisations about ways in which calculations can be made easier. By exploring the way known numbers work in situations that are familiar to them, they can reach generalisations that are best expressed using algebra and can then be used to simplify algebraic expressions. The Working Mathematically process of reflection enables students to connect algebraic operations with numerical operations.

In this article, we give some examples of what can be done with students who have already formed a sound concept of a variable. Further examples are available in sample pages from our workbook (McMaster & Mitchelmore, 2007) which are available on the 2007 Conference page of the MANSW website and at www.workingmath.net.

Figure 1. Balancing a pile of plates, saucers and teacups.

Adding and Subtracting

It is easy to find familiar contexts which show that the order in which you add a set of numbers does not matter (e.g., adding up the costs of the items on a shopping list). We prefer to use the correct terms, and say that you can *commute* (i.e. reverse) the order of the numbers and *associate* (i.e. group) them in any way you wish. The obvious way to express these observations is in the generalisations $a + b + c = c + b + a$ and $(a + b) + c = a + (b + c)$. Similar reflections lead to $a + a = 2a$ and so on.

Theoretically, it would be possible to use these results to simplify expressions like $2a + 2b + 2c + 3a + 3b + a + b + 2c$, but that would be too big a jump for students who have only just been introduced to the idea of equivalent expressions. It is more effective to again embed such simplifications in a familiar context, at least at the start. For example, the above expression might give the total weight of the plates (each weighing a grams), saucers (each b grams) and teacups (each c grams) that the clown in Figure 1 is balancing on his nose. It is now easy for students to rearrange the crockery by grouping the plates, cups and saucers together and see that the total weight is $6a + 6b + 4c$ grams. Rearranging the crockery corresponds to "collecting like terms" in the algebraic expression. And there is no doubt which way of expressing the total weight gives the easier calculation, once you find out the numerical values of a, b and c.

Before students can collect like terms in expressions which include subtractions, they need to reflect on what happens when you commute negative terms (e.g., $c - b - a = -a - b + c$), add a subtraction (e.g., $a + (b - c) = a + b - c$), subtract an addition (e.g., $a - (b + c) = a - b - c$) and subtract a subtraction (e.g., $a - (b - c) = a - b + c$). All these generalisations can be reached by reflecting on calculations in familiar contexts.

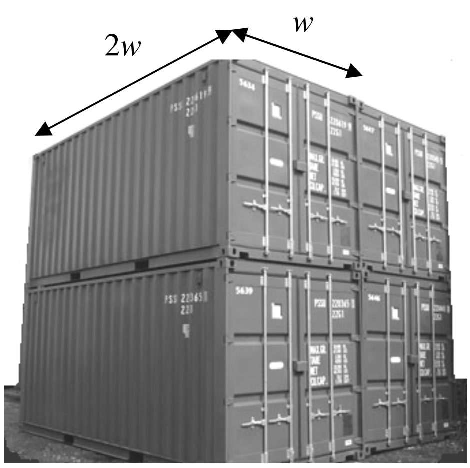

Figure 2. A pile of shipping containers.

Multiplication

As for addition, it is easy to find familiar contexts for multiplication (e.g., calculating areas and volumes). Through these, students can learn that you can commute and associate factors. The corresponding generalisations are $a \times b \times c = c \times b \times a$ and $(a \times b) \times c = a \times (b \times c)$. There are then some conventions to be learnt, in particular that of omitting the multiplication sign to show that factors are associated (e.g., $ab \times c = a \times bc$ to emphasise particular associations or $abc = cba$ to emphasise that association does not matter).

The operation that corresponds to collecting like terms for addition could be called "collecting like factors". As a first introduction, you might consider the pile of containers in Figure 2. Each container has a square cross-section and a length equal to twice the side of the square. If the side of the square is w, then the volume of each container is $w \times w \times 2w$. Since each container can be split into two cubes of side w, this expression must equal $2 \times w^3$. By commuting the factors, it can be seen that both expressions are equal to $2 \times w \times w \times w$, most simply written as $2w^3$. Similarly, the volume of the entire stack of containers is either $4 \times 2w^3$ (four containers) or $2w \times 2w \times 2w$ (looking at the dimensions of the stack) and students can explain the equivalence by showing that both are equal to $2 \times 2 \times 2 \times w \times w \times w$. Emphasising the fact that factors can be commuted and associated at will should avoid errors like $2a \times 3b = 5ab$ that are so often made by beginners.

A simple situation involving addition and multiplication is shown in Figure 3. If there are x rows of apples in a tray and y apples in each row, then each tray has xy apples. If the box contains z trays, then it must contain xyz apples. The total number of apples is therefore $2xy + xyz$. By thinking about the situation, it will become clear that $2xy$ and xyz are not like terms. We can only add these two numbers if we know the number of trays in the box.

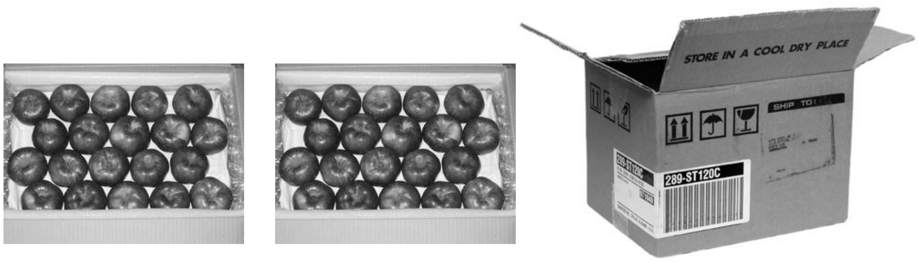

Figure 3. Trays of apples and a box of trays

Fractions and Division

The next step is to reflect on what happens when expressions involve division. In the course of a simple investigation (e.g., dividing biscuits between children), students can find that division by a number corresponds to multiplication by its inverse (e.g., $a \div b = a \times \frac{1}{b}$). Through exploring further contexts, they can then find that multiplication and division commute. For example, $a \times b \div c = a \div c \times b$ so both expressions can be written as $\frac{ab}{c}$. They can also find how to deal with the following:

- Multiplying by a division (e.g., $a \times (b \div c) = a \times b \div c = \frac{ab}{c}$)
- Dividing by a multiplication (e.g., $a \div (b \times c) = a \div b \div c = \frac{a}{bc}$)
- Dividing by a division (e.g., $a \div (b \div c) = a \div b \times c = \frac{ac}{b}$)

Exploration of a variety of contexts (including, most importantly, rates) will improve students' understanding of multiplication and division as well as provide a firm foundation for accurate algebraic manipulation. Our approach also highlights many parallels between addition/subtraction and multiplication/division. For example, have you ever noticed the similarity between $a - (b - c) = a - b + c$ and $a \div (b \div c) = a \div b \times c$? Reflecting on such similarities can lead to really deep understanding.

Further reflections

Of course, there is much more to Stage 4 Patterns and Algebra than the above examples illustrate. In particular, students need to generalise from contexts in which multiplication/division is *distributed* over addition/subtraction (e.g., $a \times (b + c) = ab + ac$ and $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$).

Once students have understood how algebraic generalisations enable them to replace an expression with an equivalent but simpler one, they are in a position to set up equations and manipulate them to solve problems they could not solve using arithmetic alone.

Reflecting on what happens with known numbers and then generalising is an essential feature of Working Mathematically. We believe that algebra approached in this way can only promote student understanding.

References

McMaster, H., & Mitchelmore, M. (2007). *Working Mathematically: Patterns & Algebra, Stage 4—Part B*. Sydney: Workingmaths.

Mitchelmore, M., & McMaster, H. (2007). Working mathematically in Patterns and Algebra. *Reflections 32*(1), 26-28.