

Working Mathematically in Patterns & Algebra

Mike Mitchelmore & Heather McMaster
Macquarie University

Superficial algebra

A number of amusing student errors have been circulating around the internet recently. The one we like best shows that $\frac{\sin x}{n}$ is equal to 6:

$$\frac{\cancel{\text{si}}x}{\cancel{n}} = \text{six}$$

We might laugh, but students often operate on the *look* of the symbols, not their meaning. Their algebraic errors are actually quite sensible. Students are just trying to follow the rules given to them—in this case, “Cancel letters if they’re the same above and below the line”.

If students don’t know what something means, then they use what they do know. Here are some more examples:

- $2 + 3x = 5x$ from “gather like terms and add the numbers together”.
- $(x + y)^2 = x^2 + y^2$ from “expand the brackets”, by analogy with $2(x + y) = 2x + 2y$.

We believe that the major problem with the standard textbook approach to algebra is that it encourages this type of superficial learning. Students are taught to manipulate algebraic symbols, but they don’t have much idea what the symbols mean. They often begin algebra by being told conventions like “In algebra we don’t use a multiplication sign” and from then on algebra becomes a series of increasingly difficult rules to remember. Because they don’t know the purpose of the manipulations, we often hear them ask: “What’s the use of algebra?”

Recent innovations such as matchstick patterns and cups and counters do not, on their own, solve the problem. Although they appear to be concrete, these approaches are actually quite abstract. The matchsticks and the cups and counters remain virtually as meaningless as “pure” algebraic symbols if students do not see how the algebra developed can apply to other contexts. Students often learn to “do” the matchstick problems and find the number of counters “hidden” in a cup, but see no relationship between these ideas (which, in most textbooks, appear in separate chapters).

In the topic of algebra, many students find “word problems” difficult, even though they mostly require only simple algebraic manipulations. Many word problems are highly contrived and there are many that students could work out without using algebra. For example: “Eight sheep have the same mass as three sheep and one cow. If the cow’s mass is 500 kg, what is the mass of the sheep?” The student must wonder at the intelligence of someone who wouldn’t think to just weigh the sheep. This type of approach to algebra does not argue well for the usefulness of the subject.

A Working Mathematically approach

Our solution is to start the study of algebra by making a big effort to give it meaning and purpose. The concept that gives algebra meaning is that of a *variable* (often called a pronumeral). Algebra provides a language for summarising general numerical *relationships*. The world is full of numerical relationships, so there are plenty of contexts to use. The simplest relationship is the linear relationship, so this is a good place to begin. A great deal can be done to introduce the basic concepts of algebra through linear relationships before abstract manipulation is introduced.

Such an approach essentially involves what we in New South Wales call Working Mathematically: asking questions about mathematical situations and experiences (*Questioning*), developing strategies to explore and solve problems (*Applying Strategies*), developing appropriate language to express mathematical ideas (*Communicating*), developing processes for exploring relationships and giving reasons to support conclusions (*Reasoning*), and reflecting on understanding and making connections and generalisations (*Reflecting*). We therefore call it a Working Mathematically approach to algebra.

An algebra workbook

Many teachers are familiar with our series of *Working Mathematically* workbooks. They are all designed to encourage active enquiry on the part of students, and they all provide a careful sequencing of concepts through activities linked directly to the various curriculum dot points. We believe workbooks have many advantages: If students have workbooks, they don't have to stick worksheets into their exercise books and they have ready access to missed lessons.

Working Mathematically in Patterns and Algebra (Stage 4, Part A) is the latest in the series. The main aim of this workbook is to help students make sense of algebra. In essence, they use algebra to represent and solve a variety of familiar problem situations. The focus is on the meaning of the linear expression $ax + b$ and its associated function and graph, but some non-linear contexts are included for the sake of contrast and to show the generality of the basic ideas. Throughout, numerical relationships and their algebraic representation are closely linked to specific contexts.

The first chapter explores a variety of mathematical patterns (in space, number, measurement and chance) without any algebra, leading to the idea of a generalisation. The second chapter then uses algebra to express generalisations about people running up and down stairs at a football stadium (a great form of exercise!). It begins with the simplest case (starting at ground level and going up by striding three stairs at a time) and then extends to many other cases. Students learn to predict a person's position after n strides in the form $an + b$, where a and b can have positive or negative values. (A negative value of b occurs when a person runs up the stairs from the change room beneath the stadium).



Some crucial terminology is now introduced. The symbol n is called a *variable*, a and b are called *coefficients*, and the relation predicting position is called a *function*. For example:

The function $R = 197 - 3n$ means:

“the stair number that Remy will be on is 197 minus 3 times the number of strides she has taken”. (p. 43)

The stage is now set to generalise this understanding. In the third chapter, a wide variety of contexts (including the well known cups and counters, tables and chairs, toothpick patterns, towels and pegs, fence posts and railings, and further realistic contexts) are explored for the linear functions implicit in them.

After a chapter introducing the number plane in a fairly standard manner, students learn how to draw graphs of linear functions. From a study of a multiplicity of contexts, students come to see from the graphs that the coefficient a represents a rate and b represents an initial value. This may seem to be too early to introduce the idea of *rate*, but it fits naturally into a conceptual introduction to linear relationships. In any case, students are already familiar with “travel graphs” from Stage 3 work.

The next chapter explores patterns in the graphs of non-linear relationships (using examples of periodic, inverse, exponential and quadratic functions) so students appreciate that there are many different types of patterns—not all linear ones.

The final chapter looks at various methods of problem solving: using trial and error, graphs and working backwards. Again, the problems are set in a variety of contexts, ending up with the traditional “guess my number” activity. The emphasis up until now has been on writing algebraic expressions and functions. Now the idea of an *equation* is introduced, but only as a method of symbolising the backtracking process. Students also learn how to use brackets to avoid confusion over the order of operations.

After using this workbook, we believe students will have a fundamental understanding of the meaning and purpose of algebraic symbolism (especially in relation to linear relationships) and be ready for a more abstract and powerful treatment of algebra in the second half of Stage 4 (normally Year 8).