The rewards and difficulties of “Working Mathematically”

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Abstract

The new NSW 7- 10 Mathematics Syllabus

For the past year, NSW schools have been teaching mathematics to students in Years 7 and 8 (students mostly at “stage 4” level) from a new syllabus. There is little change in the mathematical content of the new syllabus compared with the previous one. The major change is that the new syllabus specifies the teaching process.

The Problem Solving strand in the previous syllabus has been replaced with a strand called Working Mathematically. Working Mathematically is described as a “process strand”. Rather than being taught as a separate strand, it is to be interwoven with each of the remaining five strands, described as “content strands”. For each content strand, the syllabus lists specific activities under the heading of “Working Mathematically”. The objective is that:

> Students will develop knowledge, skills and understanding through inquiry, application of problem-solving strategies including the selection and use of appropriate technology, communication, reasoning and reflection. (Board of Studies, 2002, p. 12)

“Working Mathematically” appears to be a uniquely Australian expression used to describe what mathematicians do in real life. It includes the processes of questioning, apply strategies, communicating, reasoning and reflecting. In other words, it is a process-based approach to teaching. Process-based approaches treat mathematics as a set of interesting challenges or problems rather than as a series of methods and formulae to be learnt for examinations.

Some Working Mathematically activities listed in the syllabus can be found in textbooks but many are not, largely because Working Mathematically activities are not easy to put into a textbook format. This is particularly true for Working Mathematically within the Space and Geometry strand. For example, students in stage 4 are required to:

- interpret and make models from isometric drawings *(Communicating)*
- use dynamic geometry software to investigate angle relationships *(Applying Strategies, Reasoning)*
- recognise that similar and congruent figures are used in specific designs, architecture and art work eg works by Escher, Vasarely and Mondrian; or landscaping in European formal gardens *(Reflecting)*

To implement the new syllabus, lessons need to be prepared that use the Working Mathematically processes to teach the mathematical concepts in the content strand. This requires considerable time for thought and creativity. Having prepared the lessons, more time and effort is then required to gather the physical resources necessary. These tasks are made easier when teachers collaborate in their lesson planning and continuously refine the lessons according to their combined experience. However, this is difficult to achieve in practice. Teaching is a demanding profession and generally, insufficient time is allocated to collaborative planning and lesson development.

The Trial Workbooks

To make the Working Mathematically approach of the new syllabus easier for teachers to implement, we wrote two student workbooks (Part A and Part B) for the stage 4 Space and Geometry strand (McMaster and Mitchelmore, 2003). Working Mathematically activities additional to those specified in the syllabus, are included in the workbooks in order to satisfy every content outcome of the Space and Geometry strand.

Wherever possible, workbook activities relate to real life or other learning areas (physics, chemistry, biology, geography, surveying, sport, history, art, design and religion). The activities are also carefully sequenced to follow an abstraction process advocated by Mitchelmore and White (2000). Students begin by looking at a simple case where a concept is used, then investigate similar cases and special cases before arriving at a generalization that may sometimes be expressed as a theorem or formula. Questions in the workbooks are deliberately designed to help students make linkages with existing knowledge and experience.

Students are provided with visual stimuli and a variety of tables, grids, space for geometric constructions and paper for “cut-out” activities. Computer software used in the workbook is free to download from the internet and additional manipulative materials and activities are inexpensive and readily available from discount or department stores. The workbook also serves as notes for study and future reference.

In 2004, a trial of the workbooks began in 5 schools that were self-selected. In addition, there are about 70 teachers trialling individual activities. The purpose of the trial is to gain feedback and share experiences so that the workbooks can be improved. Teachers using the workbooks are given access to a website which includes a discussion forum and a page for teaching ideas. It is hoped that the information on this page will expand as teachers make greater use of the workbooks.

This paper reports primarily on the results of informal discussions held half way through the year with teachers from three of the trial schools - an independent school for girls (Years 7 and 8), a comprehensive state school (Years 7 and 8) and a state primary school with two opportunity classes for selected gifted and talented students (Year 6) working in stage 4. Teachers and some students were asked:

- Which activities did you find the most worthwhile and why?
- What difficulties did you experience with the Working Mathematically approach?

These questions were also put to seven of the teachers who intended to trial an occasional activity but were not committed to using the workbooks as they only had a single copy from which they photocopied activities.

The experience of an independent school for girls

This school is situated on the north shore of Sydney. Students in each year are graded into four mathematics classes. There are two activities that teachers agreed to be the most worthwhile and another that gave an unforeseen benefit. The first of these activities follows on from a series of activities in which students investigate properties of angles. The activity is called “Travel Tests”:

Form a group of three or four students.
Answer the following questions, then convince the others in your group that your answers are right.
Use the space below each question to draw a diagram if necessary.

<u>*Travel Tests*</u>
1) Andy and Bill take a ride together on a see-saw. Andy sits on one side of the see-saw, 2m from the centre. Bill sits on the other side, 1 m from the centre. With each tilt, whose side travels through the greater angle?

2) Andy and Bill take a ride together on a merry-go-round. Andy chooses to ride a horse that is near the outside edge of the merry-go-round. Bill chooses to ride a horse that is nearer to the centre. Who travels faster?

3) Andy and Bill travel with their families around Australia. They both travel along the same roads, but Andy’s family travel in a clockwise direction while Bill’s family travel anti-clockwise. Who travels further?

Teachers commented that this activity changed their role as a teacher. Instead of leading the discussion, their role became that of a facilitator. Students communicated from their own authority: “This is my diagram. This is how it works.”

The next activity considered to be particularly worthwhile, was an activity that follows one in which students gave labels to various parts of circles. The activity requires them to answer the following four questions, each relating to a separate visual image in the workbook:

1) Describe the pattern of the ripples made when you throw a small pebble into still water.

2) When you pour pancake mixture into the centre of a smooth, flat pan, why does the mixture spread out in a circle?

3) Why is an archery target round?

4) If you squash a circle (eg. the circle of a roll of cardboard) so it is half of its original height, what shape does the circle become? What happens to the amount of space inside the circle as you squash it?

Teachers liked this activity because it caused students to think about mathematical concepts and use recently acquired mathematical language to describe real life situations.

A third activity was initially mentioned by one of the teachers in relation to the difficulties experienced, but having shared his experiences with another teacher from the school, he saw the activity as being worthwhile. The activity requires students to cut up plastic straws to specified lengths, thread three pieces of straw onto a long pipe cleaner, then twist the two ends of the pipe cleaner together so that (if possible) each straw piece forms the side of a triangle. The questions are as follows:

Use a pipe cleaner and straw pieces to make each of the 5 triangles listed in the table below. Each triangle has a perimeter of 24cm.

Measure the angles of each triangle to decide whether it is an acute-angled triangle or an obtuse-angled triangle, then complete the table.

Base (cm)	***Side (cm)***	***Side (cm)***	***Type of isosceles triangle***
4	*10*	*10*	*acute-angled*
6	*9*	*9*	
8	*8*	*8*	
10	*7*	*7*	
12	*6*	*6*	

What happens to the size of the apex angle as the length of the base increases?

What is the largest apex angle you can make?

What is the longest base you can make if the perimeter of the triangle is 24cm?

State one example of three side lengths that add to 24cm and do <u>not</u> make a triangle.

Explain why the length of a side of a triangle can never be more than the sum of the lengths of the other two sides.

Some of the students insisted that they could make a triangle out of the three pieces that were 12cm, 6cm and 6cm long. They showed the teacher their “triangles”. He explained to the students that these straws placed together would not make a triangle if the pipe cleaner was not threaded through them. To avoid the argument in future, he suggested that these three lengths not be included in the table. This teacher’s class was the third of the four graded classes.

The teacher of the top class had decided to omit the activity because she believed that her students were able to visualise the situation without having to take the time to make the triangles. Also, her students had textbooks and she was concerned that parents would complain if they found that they had bought books that were never used. Therefore, she decided to switch to the textbook to teach the same concepts. The same teacher analysed results of the geometry test that followed the topic, and discovered that the third class had achieved a better result than her top class for one of the extended questions. The question asked students why they would not be able to make a triangle with the three particular side lengths. It appeared that students in the third class who had argued with their teacher, had been better able to recollect and/or understand the concept.

The major difficulty found by teachers at this school was the issue of time. They found that they had spent longer than expected teaching the Space and Geometry strand, but believed that students had gained a deeper knowledge of the subject through the connections they had made.

The experience of a comprehensive state school

This school is a small central school (years K – 10) situated in the New England district of NSW. It is classified by the Department of Education and Training as being geographically isolated and it receives a socio-economic allowance. There is only one year 7 and one year 8 mathematics class. These classes have students with mixed abilities. All high school mathematics classes are taught by the same teacher.

The teacher commented that any activity of a "hands-on" nature was worthwhile for her classes, particularly if it involved making something. The two most worthwhile activities specifically mentioned were one in which students cut out possible nets to see whether they formed cubes and another in which circles of cardboard were stapled together to make Platonic solids. Student were excited about discovering these solids for themselves and were interested in the historical connection made in the following cloze passage:

The ancient Greeks believed that everything in existence came from these solids. They each represented one of the five "elements": fire, water, earth, air and the universe.

<u>Fire</u> (dryness) was represented by the Platonic solid with the smallest volume for its surface area. The Platonic solid with the fewest vertices has the smallest volume for its surface area. This solid is the..

<u>Water</u> (wetness) was represented by the Platonic solid with the largest volume for its surface area. The Platonic solid with the largest number of vertices has the greatest volume for its surface area. This solid is the..

<u>Earth</u> was thought to be stand firmly (i.e. perpendicularly) on its base. This Platonic solid is the ....

<u>The universe</u> was represented by the Platonic solid with 12 faces because the zodiac has 12 signs. This solid is the ....

<u>Air</u> is mobile. It was represented by the Platonic solid that rotated most freely when held by 2 opposite vertices. This solid is the..

The teacher said that one student, who had lower abilities than the others in numeracy and literacy, "could not spell a word like 'school' but could spell 'tetrahedron' and could tell you about each of the platonic solids in great detail."

The "Travel Tests" activity (documented previously) was given to students as homework. Having discussed the questions with his father, one student came to class believing that his answers were right. His father (who used a circular saw analogy) had convinced him that the answer to the second question was that Andy and Bill travelled at the same speed. The teacher could not alter his conception by drawing a diagram, so she took the class into the schoolyard and had two children run around together in circles, one inside the other. The puffing and panting of the student who had run around the outer circle convinced the others that he had run faster. Class discussion over the other questions in this activity also revealed other misconceptions about distance, time and speed.

The only difficulty this teacher mentioned was that she did not have similar workbooks for the other strands of the syllabus. She said that some of her students do not work from textbooks. They like the scaffolding of answers that is provided by the workbook. Also, she found that most of her students have no interest in trying to remember anything with which they cannot make real life connections.

The experience of selective classes in a state school

This school is situated on the north shore of Sydney. Students in its four opportunity classes (two classes in year 5 and two in year 6) are selected from primary schools across the lower north shore of Sydney based largely on a test given to them in year 4. One of the teachers commented that in his class of 30 students, 11 received high distinctions in the Australasian Schools Mathematics Competition conducted by the University of NSW. The two opportunity classes have both completed Stage 3 mathematics.

The teachers said they enjoy teaching from the workbooks as they encourage higher order thinking and small group work. They found that the workbooks benefited students who were good at mathematics by requiring them to give precise explanations in words. One example of this is in the chapter on angles. Having drawn vertically opposite angles and found that when one angle becomes smaller, its vertically opposite angle becomes smaller, the question is *Why does this happen?* One of the students commented "its easier to just know it".

Another question which these students found difficult was the last of the questions in the "Travel Tests" activity (documented earlier) in which they were asked "*Who travels further?*" No one considered the practical situation of cars travelling on different sides of the road.

The students particularly liked an activity that enabled them to be artistically creative while learning about similarity:

Below are two historic Spanish Islamic design. Use the grids on this page to create your own geometrical designs. With pencils, colour polygons with the same shape the same colour.

Students were allowed to work on their designs while listening to a story being read. Their designs were mounted and displayed in the classroom. Figure 1 shows two of their many beautiful designs – one drawn on a square grid and one on an isometric grid.

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Designs by two Year 6 students

One teacher noticed that, when tested on Space and Geometry, his students all did well and there was less of a spread of results compared with past years when he had taught the same material to opportunity classes. The only difficulty he perceived was that his students would be bored if they have to repeat the mathematics they've already learnt next year when they start high school.

The experience of teachers in less-committed schools

Seven teachers interviewed were interested in trying out some activities in the workbooks but were not committed to doing so because their students had not bought workbooks. All but one of these teachers were at schools where they were the only teacher using workbook activities.

Half way through the school year, one of the seven teachers had not yet tried an activity from the workbook because Space and Geometry was not in their school's programme until later in the year. A variety of activities were chosen by the other six teachers, the most popular being the "Travel Tests" activity, activities that explored geometrical shapes using pipe cleaner and straw models, and activities that involved drawing and art. These were viewed as "fun types of extension activities". Two of the teachers reported using selected activities from the workbooks to teach students in years 9, 10 and 11. Three of the teachers reported particular incidences where a workbook activity had helped students understand a mathematical concept. These students were not in the top classes.

Four of the teachers explained that they had not used the workbook as much as they had hoped because they were using new textbooks and were more focused on trialling these. "Time pressure" was the difficulty consistently mentioned by these teachers. They had the same teaching programmes to follow that were being followed by other teachers not using the workbooks.

One teacher who liked using the workbooks, was in her first year of teaching and did not have English as her first language. She was not at a school that was trialling the workbooks. In an email about her experiences she wrote "Some teachers developed their way of teaching and are not prepared to change. Since it is my first year and I have language problem, I am more open to the idea of developing my teaching methods and use different resources than others."

Discussion

Despite large differences between trial schools in socio-economic background, gender composition, class composition and mathematical ability, there are commonalities in what their teachers perceived to be the value of a "working mathematically" approach to teaching. Teachers perceived the approach to be worthwhile because students were able to establish connections between mathematics and the real world, and because the activities required them to reason and communicate mathematical ideas.

Research has shown that when students learn mathematical concepts in real world contexts (as opposed to the traditional approach where learning is abstract or "context-free"), they demonstrate important differences in attitude and interest towards mathematics (Boaler, 1998). These differences are particularly important because they are likely to affect the students' subsequent learning of mathematics.

In all of the trial schools, students were confident enough to ask questions and communicate mathematical ideas based on their own investigations and understanding. At two of the three schools there was an indication that the working mathematically approach may be reducing the spread of test results. This could be because when students who make their own findings and communicate them, they are better able to recall the concept in tests. There is evidence from studies in England (Boaler, 1998) and in United States (Schoenfeld, 2002) that a process-based approach to teaching mathematics can narrow the performance gap between students within a class while raising the performance of the class as a whole, particularly in tests of problem-solving. However, many teachers in non-trail schools view problem-solving and the process-based approach as being better suited to higher ability classes.

There is also a perception that Working Mathematically activities take too much time. This time could be reduced as teachers and students become more familiar with the teaching approach. Activities could become more streamlined for example, by having cut straw pieces colour-coded for different lengths and stored for future lessons rather than having to be cut for each activity. However, if Working Mathematically does take more time, the question must be asked as to whether it is time well spent. If it leads to a deeper level of understanding, benefits could come by reducing the time necessary for revision when the topics are extended in later stages of the syllabus. Also, the workbooks make connections between Space and Geometry and other content strands such as the Measurement strand and the Patterns and Algebra strand. Although the time spent teaching Space and Geometry may be greater, the overall time required to teach the syllabus may not be so greatly affected.

In conclusion, it appears that there is much to be gained from Working Mathematically as it can give students a greater depth of understanding and a positive attitude towards mathematics. However, even when resources are available to support this process-based approach, many teachers still perceive that there is insufficient time available to implement it.

The introduction of the new 7–10 mathematics syllabus requires a collaborative effort. There are teachers who are not fully aware of the changes to the syllabus and believe that they are following the syllabus if they simply cover the content in the new textbooks. Others do not wish to change a teaching approach that they have used successfully for many years. This makes programming difficult for those within the same schools who are open to the change.

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